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Proof of a Conjecture for the Perturbed Gelfand Equation from Combustion Theory

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In this paper, we prove that when the domain B is a two dimensional ball, for all small positive ε, the solution curve $\{(\lambda, u)\}$ of the perturbed Gelfand equation $-\Delta u = \lambda e^{u/(1+\varepsilon u)}$ with $u|_{\partial B} = 0$ is exactly S-shaped. This settles a long-standing conjecture on the exact number of solutions to this equation.

1. INTRODUCTION

The so-called perturbed Gelfand equation

$$-\Delta u = \lambda e^{u/(1+\varepsilon u)} \text{ in } B, \qquad u = 0 \text{ on } \partial B, \tag{1}$$

plays an important role in combustion theory, where λ is a nonnegative constant, B is the unit open ball in R^n, and ε is a small positive number. We refer to [BE] for more background of this problem.

A fundamental problem is to determine the exact number of solutions for (1). It was conjectured and strongly supported by numerical evidences that when the space dimension $n = 1, 2$, for sufficiently small $\varepsilon > 0$, the solution curve $\{(\lambda, u)\}$ of (1) is exactly S-shaped: there are positive constants $\lambda_1(\varepsilon) < \lambda_2(\varepsilon)$ such that (1) has exactly 1, 2 or 3 positive solutions according as (i) $\lambda \in (0, +\infty) \setminus [\lambda_1(\varepsilon), \lambda_2(\varepsilon)]$, (ii) $\lambda = \lambda_1(\varepsilon), \lambda_2(\varepsilon)$, or (iii) $\lambda \in (\lambda_1(\varepsilon), \lambda_2(\varepsilon))$.

This problem has attracted a great deal of efforts since the 1970s. In 1974, S. V. Parter [Pa] proved that for $n = 2$ and any $\varepsilon > 0$ small, there

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exists $\underline{\lambda}_1(\varepsilon) < \bar{\lambda}_1(\varepsilon) < \underline{\lambda}_2(\varepsilon) < \bar{\lambda}_2(\varepsilon)$ such that (1) has a unique positive solution if $\lambda \in (0, \underline{\lambda}_1(\varepsilon)] \cup [\bar{\lambda}_2(\varepsilon), +\infty)$ and (1) has at least three positive solutions if $\lambda \in [\bar{\lambda}_1(\varepsilon), \underline{\lambda}_2(\varepsilon)]$. At the end of his paper, Parter states: "*Unfortunately, we have not established that there are at most three solutions. Such a result would be extremely desirable. Indeed, the computational results make it very clear that such is the case.*"

In 1980, E. N. Dancer [Da] proved, among other things (see Section 7 of [Da]), that for any small positive $\lambda_0 > 0$ and $n = 1,2$, one can find an $\varepsilon_0 > 0$ small such that if $\varepsilon \in (0, \varepsilon_0)$ then there is a constant $\lambda_2(\varepsilon) > 0$ such that (1) has exactly three positive solutions if $\lambda \in (\lambda_0, \lambda_2(\varepsilon))$, (1) has exactly two positive solutions if $\lambda = \lambda_2(\varepsilon)$ and there is a unique positive solution if $\lambda > \lambda_2(\varepsilon)$. This leaves the conjecture unsolved for the small λ-range: $0 < \lambda < \lambda_0$.

In 1985, S. P. Hastings and J. B. McLeod [HM], using quadratures, proved that the conjecture is true for the case $n = 1$. Further progress has been made recently for the $n = 1$ case. In 1994, S.-H. Wang [Wa] proved, again using quadratures, that the conjecture is true for $n = 1$ and $\varepsilon \leqslant 1/4.4967$. This upper bound for ε was improved to 1/4.35 in a recent preprint [KL] by P. Korman and Y. Li, where they use local bifurcation arguments and recently developed techniques of Korman, Li and Ouyang [KLO], together with quadratures.

We remark that these upper bounds for ε are rather sharp, since for $\varepsilon \geqslant 1/4$, it is easy to show that (1) has a unique positive solution for every $\lambda > 0$, and every integer n (see, e.g., [BIS], [CS]). Therefore the conjecture is false if $\varepsilon \geqslant 1/4$. We also want to point out that the conjecture cannot be extended to $n \geqslant 3$ as results in [Da] show that (1) can have a large number of solutions for certain values of λ when $3 \leqslant n \leqslant 9$.

In this paper, we settle this conjecture for the case $n = 2$. In fact, we prove the following result.

THEOREM 1. *Suppose that $n = 2$ and $\varepsilon > 0$ is sufficiently small, then the solution curve $\{(\lambda, u)\}$ of (1) is exactly S-shaped. Moreover, the solutions lying on the upper branch and lower branch of the solution curve are asymptotically stable, while that on the middle branch are unstable.*

Remarks.

(a) When we say the solution curve $\{(\lambda, u)\}$ is exactly S-shaped, we mean that the corresponding curve $\{(\lambda, u(0))\}$ in R^2 is exactly S-shaped. The correspondence between (λ, u) and $(\lambda, u(0))$ is 1-1.

(b) On the solution curve, there are exactly two turning points. The one with larger value of $u(0)$ will be called the upper turning point and the other one called the lower turning point. The upper branch of the solution curve refers to the part that connects the upper turning point and (∞, ∞), the lower branch refers to the part connecting $(0, 0)$ and the lower turning point, and the middle branch stands for the remaining part, which connects the two turning points.

(c) The stability of the solutions is understood in the context that the solutions are regarded as steady-state solutions of the corresponding parabolic problem. Our proof actually shows that the solutions on the middle branch have Morse index 1, i.e., the linearized eigenvalue problem has exactly one negative eigenvalue, while the other solutions (except the two turning points) have Morse index 0.

(d) Most of our arguments can be carried out for more general nonlinearities than that used in this paper, but since these particular nonlinearities are of our main interests, no attempt has been made to push our theorems in the direction of generalizations.

Our approach was partly motivated by recent works of Korman, Li, Ouyang and Shi [KLO, KLO2, KL, OS], where exact multiplicity results were proved through local bifurcation arguments. [KL] and [KLO] deal with the one dimensional case, and [KLO, OS] deal with the two or higher dimensional case but with rather different nonlinearities. The technical difficulties here are different.

We explain in the following the main ideas of our proof.

Make a change of variables $w = \varepsilon^2 u$ and $\mu = \lambda \varepsilon^2 e^{1/\varepsilon}$. Then (1) is equivalent to

$$\Delta w + \mu e^{-1/(\varepsilon + w)} = 0, \qquad w|_{\partial B} = 0. \tag{2}$$

This suggests that

$$\Delta v + \eta e^{-1/v} = 0, \qquad v|_{\partial B} = 0 \tag{3}$$

is a good approximation of (2) when ε is small.

We can prove that the solution curve $\{(\eta, v)\}$ of (3) is exactly $\subset$-shaped. In fact, we have the following result.

THEOREM 2. *Let $n = 2$. There exists $\eta_0 > 0$ such that (3) has a positive solution if and only if $\eta \geqslant \eta_0$. Moreover, there is a unique solution v_{η_0} when $\eta = \eta_0$, and there are exactly two solutions v_η^* and $v_{*\eta}$ for $\eta > \eta_0$; furthermore, $\eta \to v_\eta^*(x)$ is continuous and increasing with $v_\eta^*(0) \to \infty$ as $\eta \to \infty$; $\eta \to v_{*\eta}(0)$ is continuous and decreasing with $v_{*\eta}(0) \to 0$ as $\eta \to \infty$; $v_\eta^* \to v_{\eta_0}$, $v_{*\eta} \to v_{\eta_0}$ as $\eta \to \eta_0 + 0$.*

Indeed, making use of an observation of an explicit relationship between the solutions of (2) and (3) (see Section 2 for details), and the fact that the solution curve $\{(\eta, v)\}$ of (3) makes a "simple turn to the right" at (η_0, v_{η_0}) (see Lemma 2), we are able to use (3) to obtain an exactly $\subset$-shaped piece of solution curve of (2), denoted as Γ_0. Γ_0 connects $(\mu, w) = (\infty, \infty)$ and some (μ_0, w_0), and contains all the solutions with $\|w\|_\infty$ not small.

To understand the rest of the solution curve for (2) where $\|w\|_\infty$ is small, we combine a local bifurcation argument and Dancer's result in [Da]. Dancer's argument was based on the observation that in the form (1), the equation can be regarded as a perturbation of the well-known Gelfand equation: $-\Delta u = \lambda e^u$. Therefore, the proof of our result is based on two auxiliary equations.

Now let us explain the local bifurcation argument, which also plays an important role in proving Theorem 2 and in the analysis of Γ_0 above. To make the local bifurcation argument work, a key ingredient is the following result.

Denote $f(w) = e^{-1/(\varepsilon + w)}$, and contrary to the rest of the paper where ε is always positive, we require only $\varepsilon \geqslant 0$ here for convenience of presentation. Suppose that w is a positive solution of (2) (when $\varepsilon > 0$) or (3) (when $\varepsilon = 0$). Suppose also that w is a degenerate solution, that is, the linearization of (2) or (3) at w has zero as an eigenvalue with eigenfunction ϕ:

$$\Delta\phi + \mu f'(w)\,\phi = 0, \qquad \phi\,|_{\partial B} = 0.$$

THEOREM 3. *If $n = 2$ and $\varepsilon \geqslant 0$ (not necessarily small), then the eigenfunction ϕ defined above does not change sign in B.*

We are now ready to explain the local bifurcation argument by showing how it is used to analyze the rest of the solution curve. We can easily show that the solutions (μ, w) of (2) which are not on Γ_0 form a smooth curve that connects (μ_0, w_0) and $(0, 0)$. Let us denote it by Γ_1. If $(\mu_1, w_1) \in \Gamma_1$ is such that w_1 is a degenerate solution of (2), then by Theorem 3, a well-known local bifurcation results of Crandall and Rabinowitz [CR] can be employed to give an expression for the nearby solutions which form a piece of smooth curve; moreover, exploiting this expression of the nearby solutions and some special properties of the nonlinearity, we are able to calculate the bifurcation direction and see that near (μ_1, w_1), the solutions (μ, w) of (2) form a smooth curve lying to the left (i.e., with the same or smaller values of μ) of (μ_1, w_1). This implies that at most one such degenerate solution can exist along Γ_1. On the other hand, Dancer's result implies that at least one degenerate solution exists along Γ_1. Hence there is a unique degenerate solution on Γ_1. It follows that along Γ_1, the solutions can be locally parameterized by μ except at the unique degenerate solution where the curve makes a turn to the left. Thus Γ_1 is exactly $\supset$-shaped. Therefore, the entire solution curve $\Gamma = \Gamma_0 \cup \Gamma_1$ for (2) is exactly S-shaped as required.

We would like to point out that this kind of local bifurcation arguments turn out to be very useful in getting exact multiplicity results. In [Du], [DL] and [DL2], similar arguments were used for systems of equations, while in [KLO], [KLO2], [KL] and [OS], analogous arguments were used for single equations.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1 by assuming Theorems 2 and 3. Sections 3 and 4 are devoted to the proofs of Theorems 3 and 2 respectively.

We end this section by pointing out some of the many interesting related works not mentioned in the above. Recently, M. Mimura and K. Sakamoto [MS] studied the steady-state problem for the original combustion model on a long cylindrical domain. Under the assumption that the conjecture for the perturbed Gelfand equation is true and some other conditions, they proved that there exist stable interior layered steady-state solutions, and the profiles of these solutions are determined by the solutions of the perturbed Gelfand equation. This further emphasizes the importance of the perturbed Gelfand equation in the understanding of the combustion model. N. Mizoguchi and T. Suzuki [MSu] studied (1) with B replaced by a general domain Ω. Many earlier related results can be found from the references of this paper. We also refer to [TU] for very recent progress on the perturbed Gelfand problem (1).

2. THE PROOF OF THEOREM 1 (ASSUMING THEOREMS 2 AND 3)

In this section we prove our main result Theorem 1 by using Theorems 2 and 3.

We start with the following lemma, which is of independent interest and will also be used in Section 4 later.

Lemma 1. *Suppose $f \in C^1(R)$ and B is the unit ball in R^n, $n \geqslant 1$. Then for any given $c > 0$, the problem*

$$-\Delta u = \lambda f(u), \qquad u|_{\partial B} = 0$$

can have at most one solution (λ, u) satisfying $\lambda > 0$, $u \geqslant 0$ and $u(0) = c$.

Proof. Suppose that (λ_0, u_0) is such a solution. It suffices to show that any other such solution (λ, u) must coincide with (λ_0, u_0). By [GNN], both u and u_0 are radially symmetric. It is readily checked that $v(r) = u((\lambda_0/\lambda)^{1/2} r)$ satisfies

$$(r^{n-1} v')' + \lambda_0 r^{n-1} f(v) = 0, \qquad v(0) = c, \qquad v'(0) = 0.$$

Since u_0 satisfies the above equation with the same initial values, by uniqueness of solutions to the above initial value problem (see, Proposition 2.35 of Ni and Nussbaum [NN]), we deduce $v = u_0$. In particular, $v(r) > 0$ for $r \in [0, 1)$ and $v(1) = 0$. This implies that $\lambda = \lambda_0$ and hence $u = v = u_0$. This finishes the proof of Lemma 1. ∎

Proof of Theorem 1. By Gidas, Ni and Nirenberg [GNN], the positive solutions of (3) are radially symmetric: $v(x) = v(|x|) = v(r)$ and $v'(r) < 0$ for $r \in (0, 1]$. Similarly, the positive solutions of (2) also have these properties.

For convenience of notation later, we write

$$v_\eta^*(r) = v^*(r, \eta), \qquad v_{*\eta}(r) = v_*(r, \eta),$$

and define

$$v^*(r, \eta_0) = v_*(r, \eta_0) = v_{\eta_0}(r).$$

For any $\varepsilon \in (0, v_{\eta_0}(0))$ and any $\eta \geqslant \eta_0$, since $\partial v^*(r, \eta)/\partial r < 0$ for $r \in (0, 1]$, by Theorem 2, there exists a unique $a^* = a^*(\varepsilon, \eta) \in (0, 1)$ such that

$$v^*(a^*, \eta) = \varepsilon,$$

and $a^*(\varepsilon, \eta)$ is smooth in both variables. It is easily checked that

$$w = w^*(r, \eta, \varepsilon) \equiv v^*(a^*(\varepsilon, \eta)\, r, \eta) - \varepsilon$$

is a positive solution to (2) with

$$\mu = \mu^*(\varepsilon, \eta) \equiv [a^*(\varepsilon, \eta)]^2\, \eta.$$

For each $\varepsilon \in (0, v_{\eta_0}(0))$, by Theorem 2, there is a unique $\eta_\varepsilon > \eta_0$ such that $v_*(0, \eta_\varepsilon) = \varepsilon$. For every $\eta \in (\eta_0, \eta_\varepsilon)$, we can find a unique $a_* = a_*(\varepsilon, \eta) \in (0, 1)$ such that

$$v_*(a_*, \eta) = \varepsilon.$$

Thus

$$w = w_*(r, \eta, \varepsilon) \equiv v_*(a_*(\varepsilon, \eta)\, r, \eta) - \varepsilon$$

is another positive solution of (2) with

$$\mu = \mu_*(\varepsilon, \eta) \equiv [a_*(\varepsilon, \eta)]^2\, \eta.$$

Note that $w_*(r, \eta_\varepsilon, \varepsilon) \equiv 0$ and $a_*(\varepsilon, \eta_\varepsilon) = 0$. Hence $\mu_*(\varepsilon, \eta_\varepsilon) = 0$. Thus we know

$$\Gamma_* = \{(\mu_*(\varepsilon, \eta), w_*(\,.\,, \eta, \varepsilon)) : \eta \in [\eta_0, \eta_\varepsilon]\}$$

is a piece of solution curve of (2) that joins $(0, 0)$ and $(\mu_\varepsilon, w_\varepsilon)$, where

$$\mu_\varepsilon = \mu^*(\varepsilon, \eta_0) = \mu_*(\varepsilon, \eta_0),\ w_\varepsilon(r) = w_*(r, \eta_0, \varepsilon) = w^*(r, \eta_0, \varepsilon).$$

Similarly,

$$\Gamma^* = \{(\mu^*(\varepsilon, \eta), w^*(\,.\,, \eta, \varepsilon)) : \eta \in (\eta_0, \infty)\}$$

is a piece of solution curve for (2) that joins $(\mu_\varepsilon, w_\varepsilon)$ to (∞, ∞). Thus $\Gamma = \Gamma^* \cup \Gamma_*$ is a continuous solution curve of (2) that joins $(0, 0)$ to (∞, ∞). By Lemma 1, we see that Γ contains all the solutions of (2).

We show next that Γ is exactly S-shaped.

We first analyze the curve Γ^*. By Theorem 2 and the definition of a^*, we easily see that the function $\eta \to a^*(\varepsilon, \eta)$ is strictly increasing. Therefore, $\eta \to \mu^*(\varepsilon, \eta)$ is strictly increasing. This implies that Γ^* is a smooth curve which can be parameterized by μ.

Next we analyze Γ_*. This is much harder, and we will need the help of Theorem 3 and the assumption that ε is small.

Recall from Theorem 2 that $v_*(0, \eta)$ is decreasing to 0 as $\eta \to \infty$. Hence we can find an $\eta_1 > \eta_0$ such that $v_*(0, \eta) < 1/2$ for all $\eta > \eta_1$. Since we will now deal with Γ_* only, for simplicity of notation, we will drop the subscript * from v_*, a_*, μ_* in the following.

With these conventions, now from $v(a(\varepsilon, \eta), \eta) \equiv \varepsilon$, we obtain

$$\partial a(\varepsilon, \eta)/\partial \eta = -[\partial v(a(\varepsilon, \eta), \eta)/\partial \eta]/[\partial v(a(\varepsilon, \eta), \eta)/\partial r].$$

It follows that

$$\partial \mu(\varepsilon, \eta)/\partial \eta = a(\varepsilon, \eta)\left[a(\varepsilon, \eta) - 2\eta\,\frac{\partial v(a(\varepsilon, \eta), \eta)/\partial \eta}{\partial v(a(\varepsilon, \eta), \eta)/\partial r}\right]. \tag{4}$$

Let us now make the following observations.

(a) $\eta_\varepsilon \to \infty$ as $\varepsilon \to 0$;

(b) $a(\varepsilon, \eta) \to 1$ as $\varepsilon \to 0$ uniformly for $\eta \in [\eta_0, \eta_1]$;

(c) $\partial v(a(\varepsilon, \eta), \eta)/\partial \eta \to \partial v(1, \eta)/\partial \eta \equiv 0$ as $\varepsilon \to 0$ uniformly for $\eta \in [\eta_0 + \delta, \eta_1]$, $\forall \delta \in (0, \eta_1 - \eta_0)$;

(d) $\partial v(a(\varepsilon, \eta), \eta)/\partial r \to \partial v(1, \eta)/\partial r \leqslant c_0 < 0$ as $\varepsilon \to 0$ uniformly for $\eta \in [\eta_0, \eta_1]$.

Using these observations to (4), we deduce immediately

$$\partial\mu(\varepsilon, \eta)/\partial\eta \to 1$$

as $\varepsilon \to 0$ uniformly for $\eta \in [\eta_0 + \delta, \eta_1]$. In particular, we can find an $\varepsilon_0 > 0$ small such that $\partial\mu(\varepsilon, \eta)/\partial\eta > 0$ for all $\varepsilon \in (0, \varepsilon_0)$ and all $\eta \in [\eta_0 + \delta, \eta_1]$. This implies that

$$\Gamma_*^\delta = \{(\mu_*(\varepsilon, \eta), w_*(., \eta, \varepsilon)) : \eta \in [\eta_0 + \delta, \eta_1]\}$$

can be parameterized by μ.

Denote

$$\tilde{\Gamma}_*^\delta = \{(\mu_*(\varepsilon, \eta), w_*(., \eta, \varepsilon)) : \eta \in [\eta_0, \eta_0 + \delta]\}.$$

We can see that $\tilde{\Gamma}_*^\delta$ contains at least one degenerate solution of (2) because the curve Γ has to make a change of direction there. We are going to show that if ε is sufficiently small, then $\tilde{\Gamma}_*^\delta$ contains exactly one turning point, and it is a "turn to the right". If we can do this, then clearly

$$\Gamma_0 = \Gamma^* \cup \tilde{\Gamma}_*^\delta \cup \Gamma_*^\delta$$

is a piece of solution curve for (2) that is exactly $\subset$-shaped with one turning point on $\tilde{\Gamma}_*^\delta$.

Now we set to prove our claim on $\tilde{\Gamma}_*^\delta$. We use the local bifurcation argument mentioned in the introduction. We show that if ε is sufficiently small, near any degenerate solution $(\mu, w) \in \tilde{\Gamma}_*^\delta$, the solutions form a smooth curve which makes a "turn to the right" at (μ, w). This implies that there can be only one degenerate solution on $\tilde{\Gamma}_*^\delta$, and the curve makes a "turn to the right" at that point; this is exactly what we wanted.

To stress the ε dependence, we denote by $(\mu^\varepsilon, w^\varepsilon)$ an arbitrary degenerate solution of (2) lying on $\tilde{\Gamma}_*^\delta$. We first show that $(\mu^\varepsilon, w^\varepsilon) \to (\eta_0, v_{\eta_0})$ in $R \times C^2(\bar{B})$ as $\varepsilon \to 0$. Indeed, for any sequence $\varepsilon_n \to 0$, since $\{\mu^{\varepsilon_n}\}$ is bounded from above and below by positive constants, we may assume that $\mu^{\varepsilon_n} \to \mu^0 > 0$. Using the relationship between w and v, one checks easily that $w^{\varepsilon_n} \to w^0$ in the C^2 norm, where $w^0(r) = v_*(r, \mu^0)$, which is a positive solution of (3) with $\eta = \mu_0$. By Theorem 3, the linearized eigenvalue problem of (2) at $(\mu^\varepsilon, w^\varepsilon)$ has an eigenfunction $\phi_\varepsilon > 0$. Hence it is the principal eigenfunction and is unique if we further require $\|\phi_\varepsilon\|_\infty = 1$. Using standard elliptic regularity and compact embedding results, we can assume that, passing to a subsequence if necessary, $\phi_{\varepsilon_n} \to \phi_0$ in the C^2 norm. Then we easily see that (μ^0, w^0) is a degenerate solution to (3) with principal eigenfunction ϕ_0 for the corresponding linearized eigenvalue problem. But we know from Theorem 2 and Lemma 2 that (η_0, v_{η_0}) is the only degenerate solution of (3). Hence we necessarily have $(\mu^0, w^0) = (\eta_0, v_{\eta_0})$. The above discussion can now be summarized as: For any sequence $\varepsilon_n \to 0$, there is a subsequence $\{\varepsilon'_n\}$ such that $(\mu^{\varepsilon'_n}, w^{\varepsilon'_n}) \to (\eta_0, v_{\eta_0})$. This implies $(\mu^\varepsilon, w^\varepsilon) \to (\eta_0, v_{\eta_0})$ as $\varepsilon \to 0$. Moreover, we also have $\phi_\varepsilon \to \phi_0$.

Next we use a bifurcation result of Crandall and Rabinowitz. Set $X = C_0^{2,\alpha}(\bar{B})$, $Y = C^\alpha(\bar{B})$ and $F(\mu, w) = \Delta w + \mu f(w)$, where $f(w) = f(w, \varepsilon) = e^{-1/(\varepsilon + w)}$. It is easy to see that F is a smooth mapping from $R^+ \times X$ to Y. The partial derivative F_w at (μ, w) is given by $F_w(\mu, w)\,\phi = \Delta\phi + \mu f'(w)\,\phi$. By Theorem 3 we know that *Ker* $F_w(\mu^\varepsilon, w^\varepsilon)$ is of one dimension: in fact, *Ker* $F_w(\mu^\varepsilon, w^\varepsilon) = \mathit{span}\{\phi_\varepsilon\}$. Moreover, *codim* $F_w(\mu, w) = 1$ by the Fredholm alternative. Also

$$F_\mu(\mu^\varepsilon, w^\varepsilon) = f(w^\varepsilon) \notin \mathit{Range}\ F_w(\mu^\varepsilon, w^\varepsilon)$$

since $\int_B f(w^\varepsilon)\,\phi_\varepsilon > 0$. Therefore we can use Theorem 3.2 of [CR] to conclude the following: *Near the degenerate solution* $(\mu^\varepsilon, w^\varepsilon)$, *the solutions of* (2) *form a smooth curve* (*for each fixed small* ε)

$$(\mu_\varepsilon(s), w_\varepsilon(s)) = (\mu^\varepsilon + \tau_\varepsilon(s), w^\varepsilon + s\phi_\varepsilon + z_\varepsilon(s)), \tag{5}$$

where $s \to (\tau_\varepsilon(s), z_\varepsilon(s)) \in R \times Z_\varepsilon$ *is a smooth function near* $s = 0$ *with* $\tau_\varepsilon(0) = \tau'_\varepsilon(0) = 0$, $z_\varepsilon(0) = z'_\varepsilon(0) = 0$, *where* Z_ε *is a complement of span*$\{\phi_\varepsilon\}$ *in* X.

We now substitute the expression (5) for the solutions into equation (2), differentiate the equation with respect to s twice at $s = 0$, multiply the resulting identity with $r\phi$ and integrate it from 0 to 1 to obtain

$$\tau''_\varepsilon(0) = -\mu^\varepsilon \frac{\int_0^1 f''(w^\varepsilon)\,\phi_\varepsilon^3 r\,dr}{\int_0^1 f(w^\varepsilon)\,\phi_\varepsilon r\,dr}.$$

Letting $\varepsilon \to 0$, since $f(w) = f(w, \varepsilon) \to g(w) = e^{-1/w}$, one easily checks that the right hand side of the above identity converges to

$$-\eta_0 \frac{\int_0^1 g''(v_{\eta_0})\,\phi_0^3 r\,dr}{\int_0^1 g(v_{\eta_0})\,\phi_0 r\,dr}.$$

By Lemma 2, this quantity is positive. Hence, for all small $\varepsilon > 0$, $\tau''_\varepsilon(0) > 0$. This implies that the solution curve has a "turn to the right" at the degenerate solution $(\mu^\varepsilon, w^\varepsilon)$, provided that $\varepsilon > 0$ is small. Thus we have proved our claim on $\tilde{\Gamma}_*^\delta$, and this finished our analysis on Γ_0.

To analyze the remaining part of the solution curve, we again need Theorem 3 and some local bifurcation arguments. We also need Dancer's result mentioned in the introduction. But we would like to remark that

Dancer's result covers only part of the remaining solution curve, since it is valid only if $\mu = \lambda\varepsilon^2 e^{1/\varepsilon} \geqslant \mu_0 \equiv \lambda_0 \varepsilon^2 e^{1/\varepsilon}$, and μ_0 goes to infinity as $\varepsilon \to 0$.

Denote the remaining part of the solution curve $\Gamma \backslash \Gamma_0$ by Γ_1. By our choice of η_1, we know that any $(\mu, w) \in \Gamma_1$ satisfies $w(0) < 1/2$. If such a solution is degenerate, then the discussion leading to expression (5) can be used to conclude that, near the degenerate solution (μ, w), the solutions form a smooth curve

$$(\mu(s), w(s)) = (\mu + \tau(s), w + s\phi + z(s)),$$

where $s \to (\tau(s), z(s)) \in R \times Z$ is a smooth function near $s = 0$ with $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$, where Z is a complement of span$\{\phi\}$ in X, and $\phi \in X$ is as in Theorem 3.

As before, we can substitute this expression into (2) to obtain

$$\tau''(0) = -\mu \frac{\int_0^1 f''(w)\, \phi^3 r\, dr}{\int_0^1 f(w)\, \phi r\, dr}.$$

By Theorem 3, we may assume that $\phi(r) > 0$. Since $f''(w) > 0$ whenever $w < 1 - 2\varepsilon$, we conclude that $\tau''(0) < 0$ if $w(0) < 1/2 < 1 - 2\varepsilon$ (recall from the introduction that (1) has a unique solution for all λ if $\varepsilon \geqslant 1/4$). This implies that the solution curve makes a "turn to the left" at such a degenerate solution (μ, w). Thus, we can follow the argument described in the introduction after Theorem 3 to conclude that Γ_1 is exactly $\supset$-shaped and hence $\Gamma = \Gamma_0 \cup \Gamma_1$ is exactly S-shaped.

It remains to show the stability properties of the solutions. Let us denote the two turning points by (μ^*, w^*) and (μ_*, w_*) with $\mu^* < \mu_*$. We first consider the upper branch which consists of the part of the solution curve connecting (μ^*, w^*) and (∞, ∞). Note that this upper branch is made up of Γ^* and part of $\tilde{\Gamma}^\delta_*$ (if $(\mu_\varepsilon, w_\varepsilon)$ is not the turning point (μ^*, w^*), and which can be proved is indeed the case). As solutions on $\tilde{\Gamma}^\delta_* \backslash \{(\mu^*, w^*)\}$ are non-degenerate, and that on Γ^* can be parameterized by μ, we see that solutions on the upper branch can be parameterized by μ: $(\mu, w) = (\mu, w_\mu)$, $\mu^* < \mu < \infty$. By Lemma 2 (and Section 7) of [Da], if $\mu = \lambda\varepsilon^2 e^{1/\varepsilon} \geqslant \mu_0 \equiv \lambda_0 \varepsilon^2 e^{1/\varepsilon}$ and ε is sufficiently small, then the solution $h(r)$ to

$$h'' + h'/r + \mu f'(w_\mu)\, h = 0, \qquad h'(0) = 0, \qquad h(0) = 1$$

is positive on $[0, 1]$.

Let λ_μ denote the first eigenvalue of the linearization of (2) at w_μ. To show w_μ is asymptotically stable, it suffices to show that $\lambda_\mu > 0$. We now use the above result from [Da] to show that $\lambda_\mu > 0$ for $\mu \geqslant \mu_0$. We argue indirectly. Suppose $\lambda_\mu \leqslant 0$ for some $\mu \geqslant \mu_0$. Let $\phi > 0$ with $\|\phi\|_\infty = 1$ be the corresponding eigenfunction. By Proposition 3.3 of Lin and Ni [LN], ϕ is radially symmetric: $\phi(x) = \phi(|x|) = \phi(r)$. Thus

$$\phi'' + \phi'/r + \mu f'(w_\mu)\,\phi + \lambda_\mu \phi = 0, \qquad \phi'(0) = 0, \qquad \phi(1) = 0.$$

Since $\mu f'(w_\mu) \geqslant \mu f'(w_\mu) + \lambda_\mu$, by the Sturm comparison theorem, the solution h above must have at least one zero in $[0, 1]$. This contradiction shows that $\lambda_\mu > 0$ for $\mu \geqslant \mu_0$.

Let us now consider the case $\mu^* < \mu < \mu_0$. It suffices to show that $\lambda_\mu \neq 0$ for all such μ; indeed, this would imply $\lambda_\mu > 0$ for all $\mu > \mu^*$ since $\mu \to \lambda_\mu$ is continuous and λ_μ is positive for all large μ. Again we use an indirect argument. Suppose $\lambda_\mu = 0$ for some $\mu^* < \mu < \mu_0$. Then, we must have $(\mu, w_\mu) \in \Gamma^*$ as the other solutions on the upper branch are already known to be non-degenerate. By the same reasoning as that leading to (5), we can use the local bifurcation result of [CR] to express all the solutions of (2) near (μ, w_μ) in the form (5). This implies that we can find $\mu_n \neq \mu$, $\mu_n \to \mu$ such that

$$\left\| \frac{w_{\mu_n} - w_\mu}{\mu_n - \mu} \right\|_\infty \to \infty.$$

On the other hand, recalling $(\mu, w) \in \Gamma^*$, we know $\mu = \mu^*(\varepsilon, \eta)$ for some $\eta > \eta_0$. Then

$$\frac{w_{\mu_n} - w_\mu}{\mu_n - \mu} \to \frac{\partial w^*(\,.\,, \eta, \varepsilon)/\partial \eta}{\partial \mu^*(\varepsilon, \eta)/\partial \eta},$$

which is finite as $\partial \mu^*(\varepsilon, \eta)/\partial \eta \geqslant [a^*(\varepsilon, \eta)]^2$. This contradiction completes our proof for the stability of the solutions on the upper branch. Note that our proof also shows that all the solutions on the upper branch are non-degenerate.

The stability of the solutions on the lower branch, and the instability of the solutions on the middle branch are proved in a similar way. Here one starts from the lower turning point (μ_*, w_*) near where the stability of the solutions can be determined by Theorem 3.6 of [CR], using our previously proved fact that $\tau''(0) < 0$. Again, it reduces to show that all the solutions (except the turning points) are non-degenerate. We omit the details.

This finishes the proof of Theorem 1. ∎

3. THE PROOF OF THEOREM 3

By Proposition 3.3 of [LN], ϕ is radially symmetric: $\phi(x) = \phi(r)$. Hence

$$\phi'' + \phi'/r + \mu f'(w)\,\phi = 0, \qquad \phi'(0) = \phi(1) = 0.$$

It follows that $\phi(0) \neq 0$ (see [HK, Lemma 3.1] for a more general result). We may assume that $\phi(0) > 0$.

Let $v(r) = rw'(r) + \beta$, where β is a nonnegative constant to be determined later. (Note that we do not use the usual test function $v = rw' + \beta w$ here, this turns out to be crucial.) A direct calculation shows

$$v'' + v'/r + \mu f'(w)\,v = \beta\mu f'(w) - 2\mu f(w),$$

and

$$[r(v'\phi - v\phi')]' = r[\beta\mu f'(w) - 2\mu f(w)]\,\phi \equiv \tilde{g}(r)\,\phi. \tag{6}$$

We have

$$\begin{aligned}\tilde{g}(r) &= r\mu e^{-1/(\varepsilon + w)}(\varepsilon + w)^{-2}\,[\beta - 2(\varepsilon + w)^2]\\ &= r\mu e^{-1/(\varepsilon + w)}(\varepsilon + w)^{-2}\,g(r),\end{aligned}$$

where

$$g(r) = \beta - 2(\varepsilon + w(r))^2.$$

Clearly

$$g'(r) = -4(\varepsilon + w)\,w' > 0, \qquad \forall r \in (0, 1].$$

It is also easily seen that

$$v'(r) = -r\mu f(w(r)) < 0, \qquad \forall r \in [0, 1].$$

Let

$$h(r) = rw'(r) + 2(\varepsilon + w(r))^2.$$

Then

$$h'(r) = v'(r) - g'(r) < 0, \qquad \forall r \in (0, 1).$$

Suppose for contradiction that ϕ changes sign on B. Then we can find $r_0 \in (0, 1)$ such that

$$\phi(r) > 0, \qquad \forall r \in [0, r_0), \qquad \phi(r_0) = 0, \qquad \phi'(r_0) < 0.$$

We have two cases: (a) $h(1) \geqslant 0$, or (b) $h(1) < 0$.

In case (a), we choose $\beta = -w'(1) > 0$. Then

$$v(r) > v(1) = 0, \qquad \forall r \in [0, 1); \qquad g(r) < g(1) = -h(1) \leqslant 0, \qquad \forall r \in [0, 1).$$

We integrate (6) from $r = 0$ to $r = r_0$ to obtain

$$0 < -r_0 v(r_0)\, \phi'(r_0) = \int_0^{r_0} \tilde{g}(r)\, \phi(r)\, dr < 0,$$

a contradiction.

In case (b), since $h(0) = 2(\varepsilon + w(0))^2 > 0$, we can find a unique $r_1 \in (0, 1)$ such that $h(r_1) = 0$. Choose $\beta = -r_1 w'(r_1) = 2(\varepsilon + w(r_1))^2$. Then $g(r_1) = v(r_1) = 0$ and

$$g(r) < 0 < v(r), \qquad \forall r \in (0, r_1); \qquad g(r) > 0 > v(r), \qquad \forall r \in (r_1, 1).$$

We show that a contradiction can still be derived. We have either (i) $r_0 \leqslant r_1$, or (ii) $r_0 > r_1$.

If $r_0 \leqslant r_1$, then we integrate (6) from 0 to r_0 and arrive at a contradiction as in case (a). If $r_0 > r_1$, then we can find $r_2 \in (r_1, 1]$ such that $\phi(r_2) = 0$ and $\phi(r) < 0$ for $r \in (r_0, r_2)$. We integrate (6) now from r_0 to r_2 and obtain

$$0 < -r_2 v(r_2)\, \phi'(r_2) + r_0 v(r_0)\, \phi'(r_0) = \int_{r_0}^{r_2} \tilde{g}(r)\, \phi(r)\, dr < 0.$$

This finishes our proof. ∎

4. THE PROOF OF THEOREM 2

The proof of Theorem 2 uses some techniques of [KLO2]. The following lemma is a key ingredient.

LEMMA 2. *Suppose that v_0 is a degenerate positive solution of* (3) *with $\eta = \eta_0 > 0$. Then all solutions near (η_0, v_0) lie on a smooth curve represented by $(\eta_0 + \tau(s), v_0 + O(s))$ with s small, $\tau(0) = \tau'(0) = 0$ and $\tau''(0) > 0$.*

Proof. Set $X = C_0^{2,\alpha}(\bar{B})$, $Y = C^{\alpha}(\bar{B})$ and $G(\eta, v) = \Delta v + \eta e^{-1/v}$. It is easy to see that G is a smooth mapping from $R^+ \times X$ to Y. The partial derivative G_v at (η_0, v_0) is given by $G_v(\eta_0, v_0)\,\phi = \Delta\phi + \eta_0\, g'(v_0)\,\phi$, where $g(v) = e^{-1/v}$. By Theorem 3 we know that *Ker* $G_v(\eta_0, v_0)$ is of one dimension: in fact, there exists $\phi_0 > 0$ such that *Ker* $G_v(\eta_0, v_0) = span\{\phi_0\}$. Moreover, *codim* $G_v(\eta_0, v_0) = 1$ by the Fredholm alternative. Also

$$G_\eta(\eta_0, v_0) = g(v_0) \notin \mathit{Range}\ G_v(\eta_0, v_0)$$

since $\int_B g(v_0)\,\phi_0 > 0$. Therefore the local bifurcation result of Crandall–Rabinowitz [CR] implies that all the solutions near (η_0, v_0) lie on a smooth curve represented by $(\eta_0 + \tau(s), v_0 + O(s))$ with s small and $\tau(0) = \tau'(0) = 0$. The crucial part is to show that $\tau''(0) > 0$. As in the proof of Theorem 1, after some elementary calculations we find that

$$\tau''(0) = -\frac{\eta_0 \int_0^1 g''(v_0)\,\phi_0^3 r\,dr}{\int_0^1 g(v_0)\,\phi_0 r\,dr}.$$

Claim. $g''(v_0)\,\phi_0^2 \leqslant\not\equiv g''(v_0)\,v_0'^2$.

Since $\int_0^1 g''(v_0)\,\phi_0 v_0'^2 r\,dr = 0$ (see e.g., [KLO2]), it thus follows from this claim and the fact $\int_0^1 g(v_0)\,\phi_0 r\,dr > 0$ that $\tau''(0) > 0$. Hence it suffices to establish the above claim.

We first show that $g''(v_0(r))$ changes sign exactly once in (0, 1). Since $g''(v_0(r)) = e^{-1/v_0}(1 - 2v_0)/v_0^4$, and $v_0(r)$ is strictly decreasing, it suffices to show that $v_0(0) > \frac{1}{2}$. Suppose that $v_0(0) \leqslant 1/2$, i.e., $v_0 \leqslant \frac{1}{2}$. The equation of v_0 tells us that $\lambda_1(-\Delta - \eta(e^{-1/v_0}/v_0) = 0$. As $v_0 \leqslant \frac{1}{2}$, by comparison principle of eigenvalues we have

$$\lambda_1\left(-\Delta - \eta\,\frac{e^{-1/v_0}}{v_0^2}\right) < \lambda_1\left(-\Delta - \eta\,\frac{e^{-1/v_0}}{v_0}\right) = 0.$$

This is impossible as by Theorem 3, 0 is the principle eigenvalue of the operator $-\Delta - \eta(e^{-1/v_0}/v_0^2)$. This contradiction implies that $g''(v_0(r))$ changes sign exactly once in (0, 1), say at $r = \bar{r}$.

Next we show that $-v_0'$ and ϕ_0 intersect exactly once in (0, 1). Since $-v_0'(0) = 0$, $-v_0'(1) > 0$, $\phi_0(0) > 0$ and $\phi_0(1) = 0$, $-v_0'$ and ϕ_0 intersect at least once in (0, 1). If they intersect at least twice, we may assume that there exist $0 < r_1 < r_2 < 1$ such that $(\phi_0 + v_0')(r_1) = (\phi_0 + v_0')(r_2) = 0$, and $(\phi_0 + v_0')(r) < 0$ for $r \in (r_1, r_2)$. This in particular implies that $(\phi_0 + v_0')'$ $(r_1) \leqslant 0$ and $(\phi_0 + v_0')'$ $(r_2) \geqslant 0$. By the equations of ϕ_0 and v_0', it is easy to check that the following identity holds.

$$[rv_0'\phi_0' - r\phi_0 v_0'']' = -\frac{1}{r}\,v_0'\phi_0. \tag{7}$$

Integrating (7) from r_1 to r_2, we have

$$r_2 v_0'(r_2)(\phi_0 + v_0')'\ (r_2) - r_1 v_0'(r_1)(\phi_0 + v_0')'\ (r_1) = -\int_{r_1}^{r_2} \frac{1}{r}\, v_0' \phi_0\, dr. \quad (8)$$

However, the right hand side of (8) is positive since $v_0' < 0$ and $\phi_0 > 0$, while the left hand side of (8) is non-positive due to the facts summarized ahead of (7). This contradiction shows that $-v_0'$ and ϕ_0 intersect exactly once in (0, 1). By replacing ϕ_0 by $\mu_0 \phi_0$ for some suitable positive μ_0 if necessary, we may assume that $-v_0'$ and ϕ_0 intersect exactly at $r = \bar{r}$.

Now we can easily derive from the above that $g''(v_0)\, \phi_0^2 \leqslant \not\equiv g''(v_0)\, v_0'^2$ in $[0, 1]$. This finishes the proof of Lemma 2. ∎

Proof of Theorem 2. We first show that for η large, (3) has at least a positive solution. Let $\phi \in C_0^\infty(B)$, $\phi \geqslant 0$ and $\max_B \phi = 1$. Let $\underline{v}$ be the unique solution of $\Delta v + \phi = 0$, $v|_{\partial B} = 0$; let $\bar{v}$ be the unique solution of $\Delta v + \eta = 0$, $v|_{\partial B} = 0$. It is easy to check that for suitable large η, $\bar{v} \geqslant \underline{v}$ and they are upper-lower solutions to (3), respectively. Therefore there exists $\eta_1 > 0$ such that (3) has at least a positive solution provided that $\eta \geqslant \eta_1$. Now we can set

$$\eta_0 = \inf\{\eta > 0 : (1.1) \text{ has at least a positive solution}\}.$$

Claim. $\eta_0 > 0$. If not, there exists $\eta_i \to 0$ and v_i such that $\Delta v_i + \eta_i e^{-1/v_i} = 0$. Set $\tilde{v}_i = v_i / \|v_i\|_\infty$. Then

$$\Delta \tilde{v}_i + \eta_i \frac{e^{-1/v_i}}{v_i}\, \tilde{v}_i = 0, \qquad \tilde{v}_i|_{\partial B} = 0.$$

As $e^{-1/v_i}/v_i$ is uniformly bounded, by standard elliptic regularity, $\|\tilde{v}_i\|_{W^{2,p}} \to 0$. The Sobolev Embedding Theorem implies that $\tilde{v}_i \to 0$ uniformly. However, this is impossible as $\|\tilde{v}_i\|_\infty = 1$. This contradiction implies that $\eta_0 > 0$.

Again by standard elliptic regularity, we can further show that (3) with $\eta = \eta_0$ has at least a positive solution, and we choose one of them and denote it as v_0. We claim that v_0 must be a degenerate solution. If not, then by the Implicit Function Theorem we can show that for η less than but close to η_0, (7) has at least a positive solution, and this contradicts the definition of η_0. Since v_0 is degenerate, our Lemma 2 implies that the solutions near (η_0, v_0) form a smooth curve which turns to the right in the (η, v) plane. We may denote the upper and lower branches by v^η and v_η respectively. As long as (η, v^η) and (η, v_η) are non-degenerate, the Implicit Function Theorem ensures that we can continue to extend these two branches in the direction of increasing η, and we still denote the extensions as v^η and v_η. This process of continuation towards larger values of η for both branches may be stopped at some finite η^* by one of the following three possibilities:

(i) $\|v^{\eta_n}\|_\infty$ or $\|v_{\eta_n}\|_\infty$ goes to infinity for some $\eta_n \to \eta^* - 0$;

(ii) $\|v^{\eta_n}\|_\infty$ or $\|v_{\eta_n}\|_\infty$ goes to 0 for some $\eta_n \to \eta^* - 0$ (note that by the Harnack inequality, v^η and v_η can only lose positivity through vanishing on the entire domain);

(iii) v^{η^*} or v_{η^*} is a degenerate solution.

However, (i) cannot occur since v^{η_n} and v_{η_n} are uniformly bounded by L^p estimates and Sobolev Embedding Theorem; (ii) cannot occur either as otherwise, denoting $v_n = v^{\eta_n}$ or v_{η_n},

$$0 = \lambda_1(-\Delta - \eta_n e^{-1/v_n}/v_n) \to \lambda_1(-\Delta) > 0.$$

Finally, (iii) cannot occur. This is because, if, say, (η, v^η) becomes degenerate at $\eta = \eta^*$, then Lemma 2 tells us that all solutions near (η^*, v^{η^*}) must lie to the right side of it, which is a contradiction. Therefore we can always extend these two branches of solutions to $\eta = \infty$.

By Lemma 1, we see that $\eta \to v^\eta(0)$ and $\eta \to v_\eta(0)$ must be strictly monotone and $v^\eta(0) > v_0(0) > v_\eta(0)$ for any $\eta \in (\eta_0, \infty)$. Hence

$$\lim_{\eta \to \infty} v_\eta(0) = \alpha \in [0, v_0(0)); \qquad \lim_{\eta \to \infty} v^\eta(0) = \beta \in (v_0(0), \infty].$$

We show that $\alpha = 0$ and $\beta = \infty$. By Lemma 1, this would imply that all the positive solutions of (3) are contained in these two solution branches.

Let us first show that $\beta = \infty$. In fact we show a little more than that. An argument similar to but slightly simpler than that used in the proof of Lemma 3.4 in [KLO2] shows that $\partial v^\eta(r)/\partial \eta > 0$ for all $r \in [0, 1)$ and $\eta > \eta_0$. Hence $\eta \to v^\eta(r)$ is strictly increasing and $v^\eta(r) > v_0(r)$. It follows that

$$v^\eta(r) = (-\Delta)^{-1}[\eta e^{-1/v^\eta}] \geqslant (\eta/\eta_0)(-\Delta)^{-1}[\eta_0 e^{-1/v_0}] = (\eta/\eta_0)\, v_0(r) \to \infty$$

as $\eta \to \infty$, for any $r \in [0, 1)$.

Thus to finish the proof of Theorem 2, it remains to show $\alpha = 0$. We argue indirectly. Suppose that $\alpha > 0$. Consider the initial value problem

$$(rz')' = -re^{-1/z}, \qquad z(0) = \alpha, \qquad z'(0) = 0.$$

It is easily seen that $z'(r) < 0$ for $r \in (0, r_0)$ as long as z is positive on $(0, r_0)$. If z remains positive on $[0, \infty)$, then $z(x) = z(|x|) = z(r)$ satisfies

$\Delta z = -e^{-1/z} < 0$ on R^2 and hence is a bounded sub-harmonic function on R^2. It is well known that in such a case, $z \equiv constant$. Clearly this is impossible. Hence z has a first zero $r_0 > 0 : z(r) > 0$ in $[0, r_0)$ and $z(r_0) = 0$. By continuous dependence of the solutions on the initial values, for η^* large, the unique solution z^* of the initial value problem

$$(rz')' = -re^{-1/z}, \qquad z(0) = v_{\eta^*}(0), \qquad z'(0) = 0,$$

has a first zero r^* close to r_0. But then $v^*(r) = z^*(r^*r)$ is a solution of (3) with $v^*(0) = v_{\eta^*}(0)$ but $\eta = (r^*)^2 \to r_0^2 \neq \eta^*$ as $\eta^* \to \infty$. This contradicts Lemma 1. Hence we must have $\alpha = 0$. The proof of Theorem 2 is now complete. ∎

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REFERENCES

[BE] J. Bebernes and D. Eberly, "Mathematical Problems from Combustion Theory," Springer-Verlag, Berlin/New York, 1989.

[BIS] K. J. Brown, M. M. A. Ibrahim, and R. Shivaji, S-shaped bifurcation curves, *Nonlinear Anal.* **5** (1981), 475–486.

[CS] A. Castro and R. Shivaji, Uniqueness of positive solutions for a class of elliptic boundary value problems, *Proc. Royal Soc. Edinburgh Sect. A* **98** (1984), 267–269.

[CR] M. G. Crandall and P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, *Arch. Rational Mech. Anal.* **52** (1973), 161–180.

[Da] E. N. Dancer, On the structure of solutions of an equation in catalysis theory when a parameter is large, *J. Differential Equations* **37** (1980), 404–437.

[Du] Y. Du, Uniqueness, multiplicity and stability for positive solutions of a pair of reaction-diffusion equations, *Proc. Royal Soc. Edinburgh Sect. A* **126** (1996), 777–809.

[DL] Y. Du and Y. Lou, Some uniqueness and exact multiplicity results for a predator–prey model, *Trans. Amer. Math. Soc.* **349** (1997), 2443–2475.

[DL2] Y. Du and Y. Lou, S-shaped global bifurcation curve and Hopf bifurcation of positive solutions to a predator–prey model, *J. Differential Equations* **144** (1998), 390–440.

[GNN] B. Gidas, W.-M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.

[HM] S. P. Hastings and J. B. McLeod, The number of solutions to an equation from catalysis, *Proc. Royal Soc. Edinburgh Sect. A* **101** (1985), 15–30.

[HK] M. Holzman and H. Kielhofer, Uniqueness of global positive solution branches of nonlinear elliptic problems, *Math. Ann.* **300** (1994), 221–241.

[KL] P. Korman and Y. Li, On the exactness of an S-shaped bifurcation curve, preprint.

[KLO] P. Korman, Y. Li, and T. Ouyang, Exact multiplicity results for boundary-value problems with nonlinearities generalizing cubic, *Proc. Royal Soc. Edinburgh Sect. A* **126** (1996), 599–616.

[KLO2] P. Korman, Y. Li, and T. Ouyang, An exact multiplicity result for a class of semilinear equations, *Comm. Partial Differential Equations* **22** (1997), 661–684.

[LN] C. S. Lin and W.-M. Ni, A counterexample to the nodal domain conjecture and a related semilinear equation, *Proc. Amer. Math. Soc.* **102** (1988), 271–277.

[MS] M. Mimura and K. Sakamoto, Multi-dimensional transition layers for an exothermic reaction-diffusion system in long cylindrical domains, *J. Math. Sci. Univ. Tokyo* **3** (1996), 109–179.

[MSu] N. Mizoguchi and T. Suzuki, Equations of gas combustion: S-shaped bifurcation and mushrooms, *J. Differential Equations* **134** (1997), 183–215.

[NN] W.-M. Ni and R. D. Nussbaum, Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$, *Comm. Pure Appl. Math.* **38** (1985), 67–108.

[OS] T. Ouyang and J. Shi, Exact multiplicity of positive solutions for a class of semilinear problem, preprint.

[Pa] S. V. Parter, Solutions of a differential equation arising in chemical reaction processes, *SIAM J. Appl. Math.* **26** (1974), 687–716.

[Wa] S.-H. Wang, On S-shaped bifurcation curves, *Nonlinear Anal.* **22** (1994), 1475–1485.

[TU] K. Taira and K. Umezu, Semilinear elliptic boundary value problems in chemical reactor theory, *J. Differential Equations* **142** (1998), 434–454.